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CHARACTERIZATION OF LITACT GRAPHS

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Abstract: The litact graph of a graph $G = (V, E)$, denoted $L_{ct}(G)$, is a graph having vertex set $E(G) \cup C(G)$ in which its two vertices are adjacent if they correspond to either two adjacent edges of G or adjacent cut-vertices of G or one vertex corresponded to an edge e_i of G and other vertex corresponds to a cut-vertex c_j of G such that e_i is incident to c_j, here $C(G)$ is the set of cut-vertices of G. In this paper, we establish structural characterization of litact graphs.

Keywords and Phrases: Lict graph, Litact graph, Maximal clique.

2020 Mathematics Subject Classification: Primary: 05C75, Secondary: 05C76.

1. Introduction and Preliminaries

We refer the reader to [12] for graph theoretical terminology. In this paper, we considered only finite, simple, undirected and connected graphs. Sets $V(G)$, $E(G)$ and $C(G)$ are vertex set, edge set and cut-vertex set respectively of G. A vertex v

of graph G is called a cut-vertex if $G - v$ is a disconnected graph.

In a graph $G = (V, E)$, if $V' \subseteq V$ then a *clique* is an induced subgraph $< V' >$ of G which is a complete graph, i.e., a clique is a subgraph of G in which every two vertices are adjacent. A *maximal clique* is a clique if it is not a subgraph of larger order clique. For example, in Figure 6, graph G has four cliques G_1, G_4, G_5 and G_6 of orders ≥ 3 and two cliques K_2's.

The line graph [12] of a graph G, denoted by $L(G)$, is a graph having vertex set $E(G)$ in which two of these vertices are adjacent if corresponding edges of graph G are adjacent.

Kulli and Muddebihal [18] introduced the concept of line-cut (or, in short, lict) and litact graphs as follows:

Lict graph $L_c(G)$ and litact graph $L_{ct}(G)$ of a graph $G = (V, E)$ have vertex set $E(G) \cup C(G)$; in lict graph $L_c(G)$, two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an edge e_i of G and other vertex corresponds to a cut-vertex c_j of G such that e_i is incident with c_j and in litact graph, two vertices are adjacent if the corresponding members of G are adjacent or incident. A graph G and its $L(G)$, $L_c(G)$ and $L_{ct}(G)$ are shown in Figure 1.

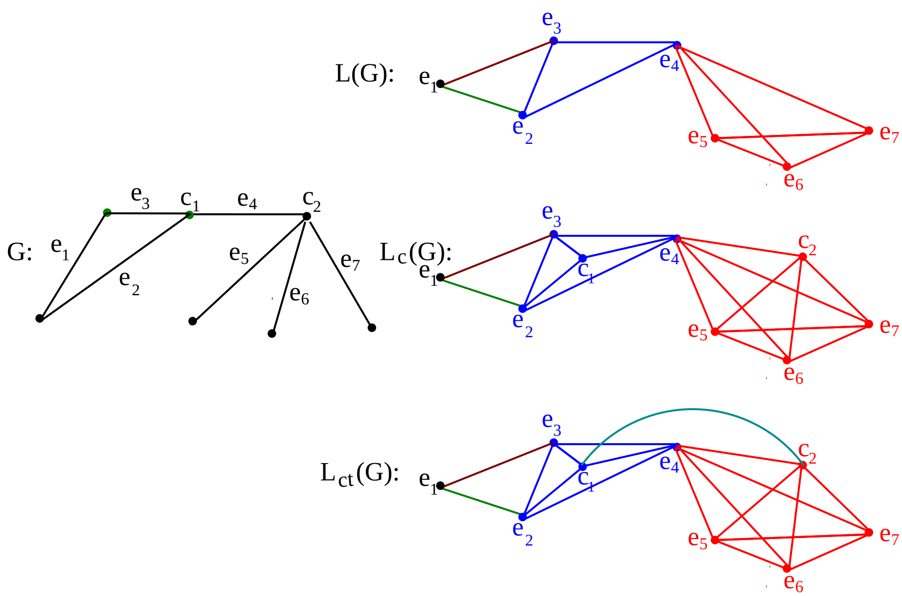

Figure 1: A graph G and its $L(G)$, $L_c(G)$ and $L_{ct}(G)$

A graph G is a line (lict or litact) graph if and only if $G \cong L(H)$ ($G \cong L_c(H)$ or $G \cong L_{ct}(H)$) for some graph H; and H is called the line (lict or litact)-root respectively of G.

The characterization of line graphs is given in the following Theorem:

Theorem 1.1. [12] *G is a line graph if and only if $E(G)$ can be partitioned into complete subgraphs such that no vertex lies in more than two of these subgraphs.*

Acharya et al. [1] renamed lict graph as line-cut graph and characterized as follows:

Theorem 1.2. *The following statements are equivalent:*

(1) *$G = (V, E)$ is a lict graph.*

(2) *The edges of G can be partitioned into cliques in such a way that no vertex lies in more than two of these cliques and for each clique G',*

  (i) *if each vertex of G' lies in two cliques of the partition then $G - E(G')$ is connected and*

  (ii) *if atleast one vertex v of G' does not lie in another clique of the partition then $G - E(G') - v$ is disconnected.*

Litact graph of any graph was introduced by Kulli and Muddebihal in [18] and till now no characterisation has been done. Motivated from characterization concept, we tackle this problem in our paper.

2. Main Results

Theorem 2.1. *The following statements are equivalent:*

(1) *A connected graph $G = (V, E)$ is a litact graph.*

(2) *The edges of G can be partitioned into maximal cliques such that for each maximal clique (say G_i) following conditions are satisfied;*

  (i) *except atmost one vertex, each vertex of G_i lies in maximum two cliques and if vertex v of G_i lies in atleast 3 cliques then all cliques other than G_i are K_2's , that is clique of size 1, whose other end vertices lie in cliques of order≥ 3 and*

  (ii) *After removing edges of K_2's cliques whose end vertices lie in maximal cliques of order ≥ 3*

    (a) *if each vertex of any maximal clique G_i lies in two cliques of partition then $G - E(G_i)$ is connected or*

(b) *if one vertex (say v) of G' does not lie in another clique then $G - E(G_i) - v$ is disconnected and*

(c) *if two adjacent cliques of order ≥ 3 (adjacent cliques means cliques having a vertex in common) have one vertex in each that lies only in these cliques, then these vertices must be adjacent in G, that is, G has no cut-vertex.*

Proof. (1) $\Rightarrow$ (2)
Suppose a connected graph G is a litact graph , that is , $G \cong L_{ct}(H)$ for some graph H. By definition of litact graphs, the edges incident on a non pendant vertex v of degree p in H, that is not a cut vertex, induces a maximal clique of order p. The edges incident on a cut vertex c of H with degree p (obviously $p \geq 2$) induce a clique of order $p+1$, i.e., order≥ 3 in G having c as a vertex of it. Two adjacent cut vertices of H induce a clique K_2 in G. We can easily observe that every edge of G is induced by either two adjacent edges of H or an edge of H whose one end vertex is a cut vertex or two adjacent cut-vertices of H. Therefore, every edge of G is contained in one maximal clique, as illustrated in Figure 2.

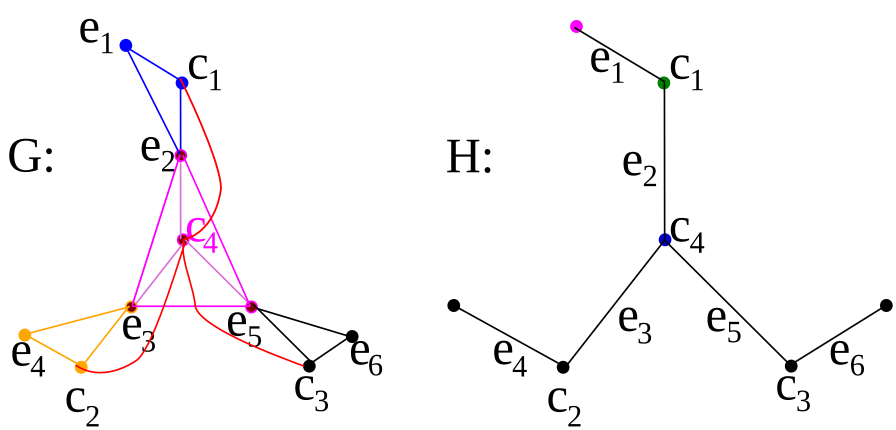

Figure 2: A graph G and a graph H such that $G \cong L_{ct}(H)$

Note that $V(G) = E(H) \cup C(H)$, where $C(H)$ is the set of cut vertices of H. Clearly if e is a pendant edge of H then the corresponding vertex in G, is contained only in one maximal clique. If e is a nonpendant edge of H then corresponding vertex in G contained in two maximal cliques. Therefore, a vertex of G induced by an edge of H does not lie in more than two maximal cliques. Also, if c_i is a cut vertex of H that is not adjacent to any other cut-vertex of H then the corresponding vertex in G is contained only in one maximal clique and if c_i is a cut vertex of H that is adjacent to m cut-vertices of H then the corresponding vertex in G is contained in m cliques K_2's also and other end vertices of these K_2's lie in cliques of orders≥ 3. Thus, except at most one vertex, each vertex of G_i lies in maximum two cliques and if vertex v of G_i lies in atleast 3 cliques then all cliques other than G_i are K_2's whose other end vertices lie in cliques of order≥ 3. Hence condition 2(i) is satisfied.

After removing edges of cliques K_2's whose end vertices lie in maximal cliques of order ≥ 3, for any maximal clique G_i of G, we complete the proof using the following two cases:

Case I. If all vertices of maximal clique G_i of G, lie in two maximal cliques then maximal clique G_i induced by those edges of graph H that incident on a vertex, say v, of graph H that is not a cut-vertex, hence every edge of H incident on a vertex v lies on a cycle, thus, each edge of maximal clique G_i correspond to two adjacent edges of H which lie on some cycle of H. Hence, if we remove all edges of maximal clique G_i of G then resultant graph $G - E(G_i)$ still remains connected, hence condition **2(ii)(a)** is satisfied. A graph shown in Figure 3 is not satisfying condition **2(ii)(a)** and this is not a litact graph.

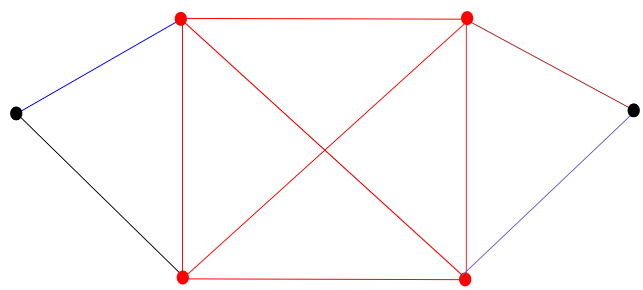

Figure 3: A graph G that is not a litact graph

Case II. If some (say m) vertices of a maximal clique G_i, lie in G_i only then this clique G_i is induced by edges incident on a cut-vertex of H. One vertex from these m vertices corresponds to a cut-vertex, say v of H and $m-1$ vertices correspond to $m-1$ pendent edges of H, incident on cut-vertex v. Since these $m-1$ pendant edges of graph H do not lie on any cycle, when we remove $E(G_i)$ and vertex v from G then resultant graph $G - E(G_i) - v$ must be disconnected. Hence Litact graph does not have pendant vertices. Thus, condition **2(ii)(b)** is satisfied. Graphs shown in Figure 4 do not satisfy condition **2(ii)(b)** and these are not litact graphs.

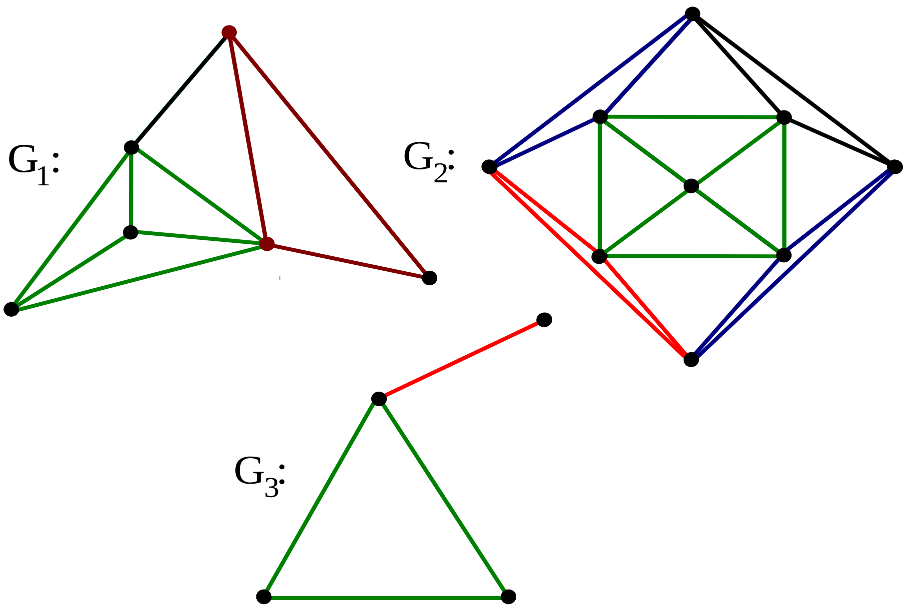

Figure 4: Graphs that are not litact graphs

Next, if some vertices of two adjacent maximal cliques (cliques in which one vertex is common are called adjacent cliques) G_i and G_j; lie only in these cliques then by Case II, these cliques are induced by two adjacent cut-vertices, hence by definition of litact graph, one vertex of G_i and one vertex of G_j must be adjacent in graph G. Thus, condition **2(ii)(c)** is satisfied. By the definition of litact graph of any graph, two adjacent cut-vertices are adjacent. Hence a litact graph has no cut-vertex. Graph shown in Figure 5 are not satisfying condition **2(ii)(c)** and these are not litact graphs.

Thus, all conditions of **(2)** are satisfied. Hence, **(1)** $\Rightarrow$ **(2)**

(2) $\Rightarrow$ **(1)**

Let for any graph G, all conditions of theorem be satisfied. Then we give construction of litact root (say H) of graph G. Let $\mathcal{P}G) = \{G_1, G_2, ..., G_n\}$ be a partition of $E(G)$ after removing edges of cliques K_2's whose end vertices lie in cliques of order≥ 3. Take $V(H) = \mathcal{P}(G) \cup U$, where U is the set of those vertices of $G - K_2$'s that lie in only one maximal clique G_i except one such vertex for each

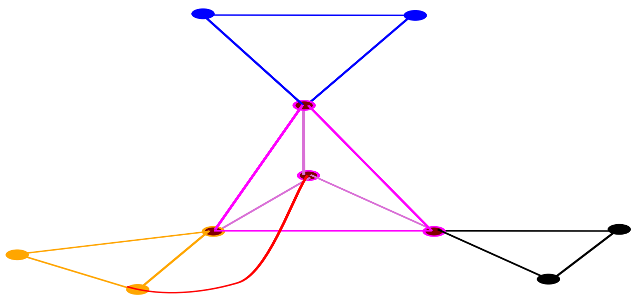

Figure 5: A graph that is not a litact graph

G_i. Then join two vertices of $V(H)$ if their intersection is non empty, i.e., graph H is the intersection graph $\Omega(\mathcal{P}(G) \cup U)$, as shown in Figure 6. For this graph H, $G \cong L_{ct}(H)$, i.e., G is a litact graph.

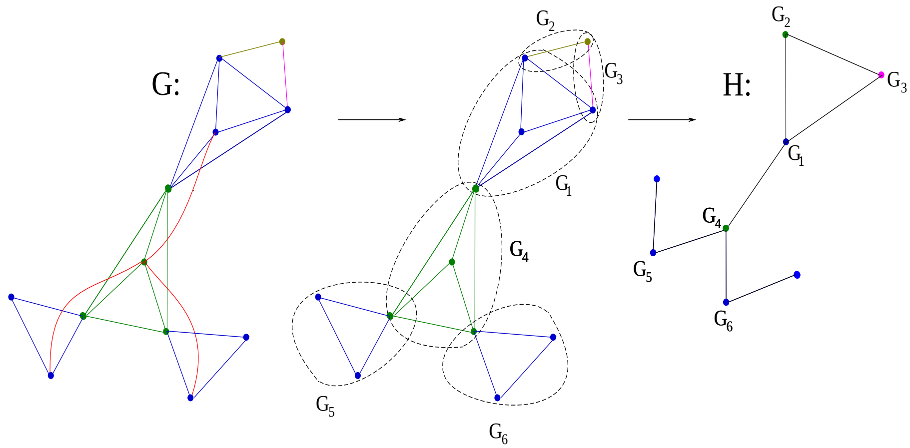

Figure 6: The construction of graph H from G such that $G \cong L_{ct}(H)$

3. Special Cases

(i) If graph G has no cut-vertex then $L(G) \cong L_c(G) \cong L_{ct}(G)$

(ii) If graph G has non adjacent cut-vertices then $L_{ct}(G) \cong L_c(G)$

4. Further Scope

Aigner [7] defined the 'line digraph' of a given digraph and Harary and Norman [13] gave a characterization of line digraphs. There are many research papers on line digraph [8, 9, 10, 11, 14, 15, 27, 19, 25]. Nagesh and Chandrasekhar [24] introduced the concept of lict digraph of a given digraph. One can think about defining Litact digraph and give characterization and many results on that. We have given many results and established characterizations of lict signed graphs $L_c(S)$, $L_{\times_c}(S)$, $L_{\bullet_c}(S)$ and also for line signed graphs $L_\times(S)$ and $L_\bullet(S)$ in [2], [3], [16], [17]. There are many research papers on line and lict graphs, line and lict sidigraphs and line and lict signed graphs as [20], [21], [22], [4], [5], [6], [26], [23]. Extension of litact graph in the realm of signed graph and sidigraphs had not been taken up. Anyone can define and characterize various types of litact digraphs, litact signed graphs and litact signed digraphs.

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